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Relativistic effective degrees of freedom and quantum statistics of neutrinos

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Analytical expressions of the relativistic effective degrees of freedom g_* with non-pure fermionic neutrinos are presented. A semi-analytical study is performed to show that g_* with pure fermionic neutrinos may be greater than g_* with pure bosonic neutrinos for non-vanishing lepton flavor asymmetries.

Keywords: Neutrino statistics; Relativistic effective degrees of freedom

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1. Introduction

We understand the neutrinos obey purely Fermi-Dirac statistics on the analogy of the electrons;¹ however, they may possess mixed statistics.^{2–8} Dolgov, et.al. studied the effects of continuous transition from Fermi-Dirac to Bose-Einstein statistics of neutrinos and discussed the possible modification of the big bang nucleosynthesis.⁹ J.I and T.K estimated the relativistic degrees of freedom with non-pure fermionic neutrinos in the early universe by numerical calculations.^{10, 11}

In this letter, to complement our previous numerical studies,^{10, 11} we show analytical expressions of the relativistic effective degrees of freedom in the early universe with non-pure fermionic neutrinos, and perform a semi-analytical study by using these analytical expressions.

2. Analytical expressions

Net number density and net energy density: The distribution function is given by⁹

$$f_i = \frac{g_i}{e^{(E-\mu_i)/T} + \kappa_i}, \quad (1)$$

where g_i , E , μ_i and T denote the number of internal degrees of freedom, energy, chemical potential and temperature, respectively. The Fermi-Bose parameter κ_i

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describes the continuous transition from Fermi-Dirac $\kappa_i = 1$ distribution to Bose-Einstein $\kappa_i = -1$ distribution via Maxwell-Boltzmann $\kappa_i = 0$ distribution.

The net number density of particle species i is obtained as^{12,13}

$$\begin{aligned} n_{i-\bar{i}} &= n_i - n_{\bar{i}} \\ &= \frac{g_i}{2\pi^2} \int_{m_i}^{\infty} E(E^2 - m_i^2)^{1/2} [f_i(E) - f_{\bar{i}}(E)] dE \\ &\simeq \frac{2g_i T^3 \xi_i}{\pi^2} F_2(\kappa_i) \end{aligned} \quad (2)$$

where

$$\xi_i = \frac{\mu_i}{T}, \quad F_s(x) \equiv \frac{\text{Li}_s(-x)}{-x}, \quad (3)$$

and $\text{Li}_s(x)$ denotes the polylogarithm function. We note that $F_2(1) = \pi^2/12$, $F_4(1) = 7\pi^4/720$ are obtained for pure fermions and $F_2(-1) = \pi^2/6$, $F_4(-1) = \pi^4/90$ are obtained for pure bosons.

Similarly, the net energy density is obtained as

$$\rho_{i+\bar{i}} = \rho_i + \rho_{\bar{i}} \simeq \frac{3g_i T^4}{\pi^2} (2F_4(\kappa_i) + \xi_i^2 F_2(\kappa_i)). \quad (4)$$

Chemical potentials: The beta-decay of down quark via weak interactions $d \rightarrow u + \ell + \bar{\nu}_\ell$ provides

$$\mu_u + \mu_\ell = \mu_d + \mu_{\nu_\ell}, \quad (\ell = e, \mu, \tau), \quad (5)$$

and the relation of $\mu_i = \mu_{\bar{i}}$ and $\mu_\gamma = \mu_W = \mu_Z = \mu_g = \mu_H = 0$ are appropriate. With the following assumptions¹⁴

$$\mu_u = \mu_c = \mu_t, \quad \mu_d = \mu_s = \mu_b, \quad (6)$$

there are only five independent chemical potentials and we take these as $\mu_u, \mu_d, \mu_{\nu_e}, \mu_{\nu_\mu}, \mu_{\nu_\tau}$. These five independent chemical potentials are uniquely determined by the following five conservation laws¹⁴

$$sQ = - \sum_{i=e,\mu,\tau} n_{i-\bar{i}} + \frac{2}{3} \sum_{i=u,c,t} n_{i-\bar{i}} - \frac{1}{3} \sum_{i=d,s,b} n_{i-\bar{i}}, \quad (7)$$

$$sB = \frac{1}{3} \sum_{i=\text{quarks}} n_{i-\bar{i}}, \quad (8)$$

$$sL_\ell = n_{\ell-\bar{\ell}} + n_{\nu_\ell-\bar{\nu}_\ell}, \quad (9)$$

where Q , B and L_ℓ denote electric charge, baryon number and lepton flavor number of the universe, respectively.

For electrically neutral universe, $Q = 0$, from Eqs. (2), (7), (8) and (9), the following coupled equations for chemical potentials up to $\mathcal{O}(\xi_i^2)$ are obtained

$$\begin{aligned} 0 &= -(\xi_e + \xi_\mu + \xi_\tau) + 4\xi_u - 3\xi_d, \\ \frac{3sB}{T^3} &= 2\xi_u + 3\xi_d, \\ \frac{3sL_\ell}{T^3} &= \xi_\ell + \frac{6}{\pi^2} F_2(\kappa_\nu) \xi_{\nu_\ell}, \end{aligned} \quad (10)$$

where we assume $m_W < T < m_t$, $\kappa_\nu = \kappa_{\nu_e} = \kappa_{\nu_\mu} = \kappa_{\nu_\tau}$ and take $\kappa_i = 1$ for the fermions in the standard model except neutrinos. From Eqs.(5) and (10), the five independent ξ_i ($i = u, d, \nu_e, \nu_\mu, \nu_\tau$) as well as ξ_ℓ are analytically determined

$$\begin{aligned} \xi_u &= \frac{s}{2T^3} \frac{1}{1 + \frac{11}{\pi^2} F_2(\kappa_\nu)} \left\{ L + \left[1 + \frac{12}{\pi^2} F_2(\kappa_\nu) \right] B \right\}, \\ \xi_d &= \frac{s}{3T^3} \frac{1}{1 + \frac{11}{\pi^2} F_2(\kappa_\nu)} \left\{ -L + \left[2 + \frac{21}{\pi^2} F_2(\kappa_\nu) \right] B \right\}, \\ \xi_{\nu_\ell} &= \frac{s}{T^3} \frac{1}{1 + \frac{6}{\pi^2} F_2(\kappa_\nu)} \left\{ 3L_\ell + \frac{1}{6(1 + \frac{11}{\pi^2} F_2(\kappa_\nu))} \left[5L - \left(1 + \frac{6}{\pi^2} F_2(\kappa_\nu) \right) B \right] \right\}, \\ \xi_\ell &= \frac{s}{T^3} \frac{1}{1 + \frac{6}{\pi^2} F_2(\kappa_\nu)} \left\{ 3L_\ell - \frac{F_2(\kappa_\nu)}{\pi^2(1 + \frac{11}{\pi^2} F_2(\kappa_\nu))} \left[5L - \left(1 + \frac{6}{\pi^2} F_2(\kappa_\nu) \right) B \right] \right\}, \end{aligned} \quad (11)$$

where $L = \sum_{\ell=e,\mu,\tau} L_\ell$.

Relativistic effective degrees of freedom: The relativistic effective degrees of freedom for energy density, g_* , is defined by^{13,14}

$$\rho = \sum_i \rho_i = \frac{\pi^2 T^4}{30} g_*. \quad (12)$$

Similarly, for entropy density, g_{*s} is defined by $s = \sum_i s_i = \frac{2\pi^2 T^3}{45} g_{*s}$. From Eqs.(12), (4) and (11), the relativistic effective degrees of freedom for energy density is obtained as

$$g_*(\xi, \kappa_\nu) = g_*(\xi_i = 0, \kappa_\nu = 1) + \Delta g_*, \quad (13)$$

where

$$g_*(\xi_i = 0, \kappa_\nu = 1) = \sum_{b=\text{bosons}} g_b + \frac{7}{8} \sum_{f=\text{fermions}} g_f, \quad (14)$$

denotes the well-known relativistic effective degrees of freedom for vanishing chemical potentials ($\xi_i = 0$) and for pure fermionic neutrinos ($\kappa_\nu = 1$).¹³ The effects on g_* from the non-vanishing chemical potentials and non-pure fermionic neutrinos are estimated as

$$\begin{aligned} \Delta g_* &= \frac{15}{4\pi^2} \sum_{i=\text{fermions} \neq \nu} g_i \xi_i^2(\kappa_\nu) + \sum_{i=\nu} \frac{45}{\pi^4} F_2(\kappa_\nu) g_\nu \xi_i^2(\kappa_\nu) \\ &\quad + \sum_{i=\nu} g_i \left(\frac{90}{\pi^4} F_4(\kappa_\nu) - \frac{7}{8} \right). \end{aligned} \quad (15)$$

Similarly, the relativistic effective degrees of freedom for entropy density is expressed as $g_{*s}(\xi, \kappa_\nu) = g_{*s}(\xi_i = 0, \kappa_\nu = 1) + \Delta g_{*s}$. For $T \sim 100\text{GeV}$, we obtain $g_{*s}(\xi_i = 0, \kappa_\nu = 1) = g_*(\xi_i = 0, \kappa_\nu = 1)$ and $\Delta g_{*s} = \Delta g_*$.

3. Discussions and summary

With the vanishing chemical potential, the equilibrium energy density of pure bosonic particle is larger than it of pure fermionic particle.¹³ One may expect that the relation $g_*^{\text{FD}} < g_*^{\text{BE}}$ is guaranteed where g_*^{FD} denotes g_* with pure fermionic neutrinos and g_*^{BE} denotes g_* with pure bosonic neutrinos. However, in our previous numerical studies,^{10,11} we have shown that this relation is not always satisfied with non-vanishing lepton flavor asymmetries in the early universe.¹⁵⁻²²

We complement our previous numerical studies^{10,11} by semi-analytical calculations. From Eqs. (11) and (15), we obtain

$$\Delta g_*(\kappa_\nu = 1) = \frac{50}{529\pi^2} \left(\frac{s}{T^3} \right)^2 \left(873B^2 - 162BL + 362L^2 + 1587 \sum_\ell L_\ell^2 \right), \quad (16)$$

for pure fermionic neutrinos, and

$$\Delta g_*(\kappa_\nu = -1) = \frac{45}{289\pi^2} \left(\frac{s}{T^3} \right)^2 \left(529B^2 - 78BL + 140L^2 + 867 \sum_\ell L_\ell^2 \right) + \frac{3}{4}, \quad (17)$$

for pure bosonic neutrinos. For the sake of simplicity, we assume $L_\ell = L/3$ and $L \gg B \sim 0$, and use very rough estimation of $s/T^3 = 2\pi^2 g_{*s}/45 = 2\pi^2 g_*/45 \sim 44$ with $T = 100\text{ GeV}$. In this case, we obtain

$$\Delta g_*(\kappa_\nu = 1) \simeq 1.65 \times 10^4 L^2, \quad \Delta g_*(\kappa_\nu = -1) \simeq 1.31 \times 10^4 L^2 + \frac{3}{4}. \quad (18)$$

and $\Delta g_*(\kappa_\nu = 1) \gtrsim \Delta g_*(\kappa_\nu = -1)$ as well as $g_*^{\text{FD}} \gtrsim g_*^{\text{BE}}$ for $L \gtrsim 0.015$.

We comment on possible application of our results in a cosmological context. In the leptogenesis scenario,²³ the baryon-photon ratio in the universe η_B is related to the lepton asymmetry Y_L via $\eta_B \propto Y_L \propto g_*^{-1}$. Thus, the lepton number in the early universe yields change of the baryon-photon ratio. More detailed analysis will be found in our future study.

In summary, analytical expressions of the relativistic effective degrees of freedom with non-pure fermionic neutrinos are presented. A semi-analytical study has been performed to complement our previous numerical studies which show that the relation of $g_*^{\text{FD}} \gtrsim g_*^{\text{BE}}$ may be allowed with non-vanishing lepton flavor asymmetries.

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